No positive roots

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The following theorem appears as Exercise 3(a) in Chapter 3 of [4], which covers random walks. I solved the exercise while reading the book in August 2025, though it took me quite a while. These notes present my argument, with the aim of using it as a test case for AI-assisted formalization and verification in Lean.

Theorem 1. Let $p \in \mathbb{R}[x]$ be a nonzero polynomial. The following conditions are equivalent:

1. There exists a nonzero polynomial $q \in \mathbb{R}[x]$ such that all coefficients of pq are nonnegative.
2. The polynomial p has no positive real root, i.e., $p(\alpha) \neq 0$ for every $\alpha > 0$.

The implication "⇒" is immediate: a nonzero polynomial with nonnegative coefficients has no positive real root, and any root of p is also a root of pq. It therefore remains to prove the implication "⇐".

If p is constant, we may simply take $q = 1/p$. Otherwise, we can write $p = cp_1 \cdots p_n$, where $c \in \mathbb{R} \setminus \{0\}$ and each p_i is a monic irreducible polynomial of degree one or two. If p has no positive real root, then neither does any of its irreducible factors.

The product of polynomials with nonnegative coefficients again has nonnegative coefficients. Thus, if for each i we can find a nonzero polynomial q_i such that $p_i q_i$ has nonnegative coefficients, then $q = c^{-1} q_1 \cdots q_n$ satisfies $pq = p_1 q_1 \cdots p_n q_n$, which has nonnegative coefficients. It therefore suffices to prove the claim for monic irreducible polynomials of degree one or two.

The linear case is immediate. Let $p = x + a$. Its unique root is $-a$, so p has no positive real root if and only if $a \geq 0$. Thus, all coefficients of p are already nonnegative, and we may take $q = 1$.

For the quadratic case, consider a monic irreducible polynomial $p = x^2 + bx + a \in \mathbb{R}[x]$. Since p is irreducible over $\mathbb{R}$, its discriminant satisfies $\Delta = b^2 - 4a < 0$. In particular, $0 \leq b^2 < 4a$, so $a > 0$. If $b \geq 0$, then p already has nonnegative coefficients, and we may take $q = 1$. It therefore remains to prove the following lemma.

Lemma 2. Let $p = x^2 + bx + a \in \mathbb{R}[x]$ satisfy $a > 0$, $b < 0$, and $b^2 - 4a < 0$. Then there exists a nonzero polynomial $q \in \mathbb{R}[x]$ such that all coefficients of pq are nonnegative.

We define a real sequence $(y_k)_{k \geq 0}$ by setting $y_0 = 0$ and requiring the formal power series identity

$$\frac{a}{x^2 + bx + a} = \sum_{k=1}^{\infty} y_k x^{k-1}.$$

Since $a \neq 0$, this uniquely determines the sequence. Comparing coefficients, we obtain the linear recurrence

$$y_0 = 0, \qquad y_1 = 1, \qquad y_{k+2} = -\frac{b}{a}\, y_{k+1} - \frac{1}{a}\, y_k \quad (k \geq 0).$$

For each $n \geq 1$, define the polynomial

$$q_n(x) = \sum_{k=1}^{n} y_k x^{k-1}.$$

Lemma 3. For every $n \geq 1$, we have $(a + bx + x^2)q_n(x) = a - ay_{n+1}x^n + y_n x^{n+1}$.

Proof. We argue by induction on n. For $n = 1$, we have $q_1 = 1$, $y_1 = 1$, and $y_2 = -b/a$. Hence

$$(a + bx + x^2)q_1(x) = a + bx + x^2 = a - ay_2 x + y_1 x^2.$$

Now suppose that the identity holds for some $n \geq 1$. Since

$$q_{n+1}(x) = q_n(x) + y_{n+1}x^n,$$

the induction hypothesis gives

$$\begin{aligned}(a + bx + x^2)q_{n+1}(x) &= a - ay_{n+1}x^n + y_n x^{n+1} + (a + bx + x^2)y_{n+1}x^n \\ &= a + (y_n + by_{n+1})x^{n+1} + y_{n+1}x^{n+2}.\end{aligned}$$

The recurrence relation implies

$$y_n + by_{n+1} = -ay_{n+2}.$$

Consequently,

$$(a + bx + x^2)q_{n+1}(x) = a - ay_{n+2}x^{n+1} + y_{n+1}x^{n+2},$$

which completes the induction. □

To prove Lemma 2, it therefore suffices to show that $y_k \leq 0$ for some $k \geq 1$. Indeed, choose n such that y_{n+1} is the first nonpositive term after y_0. Since $y_1 = 1$, we have $n \geq 1$ and $y_n > 0$. The identity above then shows that all coefficients of pq_n are nonnegative. Moreover, $q_n \neq 0$, since its constant coefficient is $y_1 = 1$.

It remains to establish the following lemma.

Lemma 4. There exists an integer $k \geq 1$ such that $y_k \leq 0$.

Proof. The characteristic polynomial of the recurrence is

$$\chi(t) = t^2 + \frac{b}{a}\,t + \frac{1}{a}.$$

Its discriminant satisfies

$$\Delta' = \frac{b^2 - 4a}{a^2} < 0,$$

so its roots are distinct and complex conjugates:

$$\lambda_{1,2} = \frac{-b \pm i\sqrt{4a - b^2}}{2a}.$$

Since $a > 0$ and $b < 0$, we have

$$\mathrm{Re}(\lambda_1) = \frac{-b}{2a} > 0, \qquad \mathrm{Im}(\lambda_1) = \frac{\sqrt{4a - b^2}}{2a} > 0.$$

Therefore, we may write

$$\lambda_1 = re^{i\phi}, \qquad \lambda_2 = re^{-i\phi}$$

for some $r > 0$ and $0 < \phi < \pi/2$.

The general solution of the recurrence has the form

$$y_k = c\lambda_1^k + d\lambda_2^k \qquad (k \geq 0)$$

for some $c, d \in \mathbb{C}$. The initial conditions $y_0 = 0$ and $y_1 = 1$ give

$$c + d = 0, \qquad c\lambda_1 + d\lambda_2 = 1.$$

Consequently, $d = -c$ and

$$c = \frac{1}{\lambda_1 - \lambda_2} = \frac{1}{2i\ \mathrm{Im}(\lambda_1)}.$$

Using Euler's formula, we obtain

$$y_k = \frac{\lambda_1^k - \lambda_2^k}{2i\ \mathrm{Im}(\lambda_1)} = \frac{r^k}{\mathrm{Im}(\lambda_1)} \sin(k\phi).$$

By Lemma 5, there exists an integer $k \geq 1$ such that $\sin(k\phi) \leq 0$. Since $r > 0$ and $\mathrm{Im}(\lambda_1) > 0$, it follows that $y_k \leq 0$. □

Lemma 5. For every $\phi \in \mathbb{R}$, there exists an integer $k \geq 1$ such that $\sin(k\phi) \leq 0$.

Proof. If $\sin\phi \leq 0$, take $k = 1$. Otherwise, by periodicity, we may replace ϕ by its representative $\theta \in (0, \pi)$ modulo 2π. Set $k = \left\lceil \frac{\pi}{\theta} \right\rceil$. Then $\pi \leq k\theta < \pi + \theta < 2\pi$, so $\sin(k\phi) = \sin(k\theta) \leq 0$. □

Example 6. Consider the polynomial $p(x) = x^2 - \frac{19}{10}x + 5$. Its discriminant is

$$\Delta = \left(\frac{19}{10}\right)^2 - 20 = -\frac{1639}{100} < 0.$$

The associated sequence is given by

$$y_0 = 0, \qquad y_1 = 1, \qquad y_{k+2} = \frac{19}{50} y_{k+1} - \frac{1}{5} y_k \quad (k \geq 0).$$

In particular,

$$y_2 = \frac{19}{50}, \qquad y_3 = \left(\frac{19}{50}\right)^2 - \frac{1}{5} = -\frac{139}{2500}.$$

Thus, y_3 is the first nonpositive term after y_0. We therefore take

$$q(x) = q_2(x) = 1 + \frac{19}{50}x.$$

We have

$$p(x)q(x) = 5 - 5y_3x^2 + y_2x^3 = 5 + \frac{139}{500}x^2 + \frac{19}{50}x^3,$$

which has nonnegative coefficients.

Remark 7 (Historical note)**.** Essentially the same construction appears in Meissner's 1911 paper, where it is attributed to Hurwitz's 1904 seminar; see [5, Introduction and Footnote 2]. For the interpretation via truncated reciprocal power series, see [1, Section 3].

In [2], Pólya proves the statement and extends it to the multivariate case. In particular, he shows that $\pm(1 + x)^N p(x)$ has nonnegative coefficients for all sufficiently large N. Explicit bounds on N were later obtained, for example, in [3].

References

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